Exemplos:

1) $f(z_1, z_2) = z_1^a z_2^b$; $a, b > 0$

$q^*(w_1, w_2, p) = ?$ $\quad \pi^* = (w_1, w_2, p) = ?$

$z_1^*(w_1, w_2, p) = ?$

$z_2^*(w_1, w_2, p) = ?$

$PMg_1 = \frac{w_1}{p} \Rightarrow a z_1^{a-1} z_2^b = \frac{w_1}{p}$ (I)

$PMg_2 = \frac{w_2}{p} \Rightarrow b z_1^a z_2^{b-1} = \frac{w_2}{p}$

$\frac{a}{b} \frac{z_2}{z_1} = \frac{w_1}{w_2} \Rightarrow z_2 = \frac{b}{a} \frac{w_1}{w_2} z_1$ subst. em (I)

$a z_1^{a-1} \left(\frac{b}{a} \frac{w_1}{w_2} z_1\right)^b = \frac{w_1}{p} \Rightarrow a^{1-b} z_1^{a+b-1} b^b \left(\frac{w_1}{w_2}\right)^b = \frac{w_1}{p} \Rightarrow z_1^{1-a-b} = \left(\frac{a}{w_1}\right)^{1-b} \cdot \left(\frac{b}{w_2}\right)^b p$

$z_1^*(w_1, w_2, p) = \left(\frac{a}{w_1}\right)^{\frac{1-b}{1-a-b}} \left(\frac{b}{w_2}\right)^{\frac{b}{1-a-b}} p^{\frac{1}{1-a-b}}$ $\quad z_2^*(w_1, w_2, p) = \left(\frac{a}{w_1}\right)^{\frac{a}{1-a-b}} \left(\frac{b}{w_2}\right)^{\frac{1-a}{1-a-b}} p^{\frac{1}{1-a-b}}$

$q^* = f(z_1^*, z_2^*) = \left[\left(\frac{a}{w_1}\right)^{\frac{1-b}{1-a-b}} \left(\frac{b}{w_2}\right)^{\frac{b}{1-a-b}} p^{\frac{1}{1-a-b}}\right]^a \times \left[\left(\frac{a}{w_1}\right)^{\frac{a}{1-a-b}} \left(\frac{b}{w_2}\right)^{\frac{1-a}{1-a-b}} p^{\frac{1}{1-a-b}}\right]^b$

$q^* = \left(\frac{a}{w_1}\right)^{\frac{a-ba}{1-a-b}} \left(\frac{b}{w_2}\right)^{\frac{ba}{1-a-b}} p^{\frac{a}{1-a-b}} \times \left(\frac{a}{w_1}\right)^{\frac{ab}{1-a-b}} \times \left(\frac{b}{w_2}\right)^{\frac{b-ba}{1-a-b}} p^{\frac{b}{1-a-b}} = p^{\frac{a+b}{1-a-b}} \left(\frac{a}{w_1}\right)^{\frac{a}{1-a-b}} \left(\frac{b}{w_2}\right)^{\frac{b}{1-a-b}}$

$$\boxed{q^*(w_1, w_2, p) = p^{\frac{a+b}{1-a-b}} \left(\frac{a}{w_1}\right)^{\frac{a}{1-a-b}} \left(\frac{b}{w_2}\right)^{\frac{b}{1-a-b}}}$$

$$1 + \frac{a}{1-a-b} = \frac{1-a-b+a}{1-a-b}$$

$$w_1^{-\left[\frac{a}{1-a-b}+1\right]} = w_1^{-\frac{1-b}{1-a-b}} \qquad a^{\frac{a}{1-a-b}} \cdot a = a^{\frac{1-b}{1-a-b}}$$

$$\boxed{z_1^*(w_1, w_2, p) = p^{\frac{1}{1-a-b}} \left(\frac{a}{w_1}\right)^{\frac{1-b}{1-a-b}} \left(\frac{b}{w_2}\right)^{\frac{b}{1-a-b}}}$$

$$\boxed{z_2^*(w_1, w_2, p) = p^{\frac{1}{1-a-b}} \left(\frac{a}{w_1}\right)^{\frac{a}{1-a-b}} \left(\frac{b}{w_2}\right)^{\frac{1-a}{1-a-b}}}$$

$$\pi^* = p \cdot q^* - w_1 z_1^* - w_2 z_2^* = p^{\frac{1}{1-a-b}} \left(\frac{a}{w_1}\right)^{\frac{a}{1-a-b}} \left(\frac{b}{w_2}\right)^{\frac{b}{1-a-b}} - p^{\frac{1}{1-a-b}} \frac{a^{\frac{1-b}{1-a-b}}}{w_1^{\frac{a}{1-a-b}}} \left(\frac{b}{w_2}\right)^{\frac{b}{1-a-b}} - p^{\frac{1}{1-a-b}} \left(\frac{a}{w_1}\right)^{\frac{a}{1-a-b}} \frac{b^{\frac{1-a}{1-a-b}}}{w_2^{\frac{b}{1-a-b}}}$$

$$\boxed{\pi^* = p^{\frac{1}{1-a-b}} \left(\frac{a}{w_1}\right)^{\frac{a}{1-a-b}} \left(\frac{b}{w_2}\right)^{\frac{b}{1-a-b}} \left[1 - a - b\right]}$$

$$\frac{1}{1-a-b} - 1 = \frac{1-(1-a-b)}{1-a-b} = \frac{a+b}{1-a-b}$$

$$\frac{\partial \pi^*}{\partial p} = \cancel{\frac{1}{1-a-b}}\, p^{\frac{1}{1-a-b}-1} \left(\frac{a}{w_1}\right)^{\frac{a}{1-a-b}} \left(\frac{b}{w_2}\right)^{\frac{b}{1-a-b}} \left[\cancel{1-a-b}\right] = \underbrace{p^{\frac{a+b}{1-a-b}} \left(\frac{a}{w_1}\right)^{\frac{a}{1-a-b}} \left(\frac{b}{w_2}\right)^{\frac{b}{1-a-b}}} = q^*(w_1, w_2, p)$$

$$\frac{\partial \pi^*}{\partial w_1} = -\frac{a}{\cancel{1-a-b}}\, p^{\frac{1}{1-a-b}}\, w_1^{-\frac{a}{1-a-b}-1}\, a^{\frac{a}{1-a-b}} \left(\frac{b}{w_2}\right)^{\frac{b}{1-a-b}} \left[\cancel{1-a-b}\right] = -p^{\frac{1}{1-a-b}} \left(\frac{a}{w_1}\right)^{\frac{1-b}{1-a-b}} \left(\frac{b}{w_2}\right)^{\frac{b}{1-a-b}} = -z_1^*(w_1, w_2, p)$$

$$\frac{\partial \pi^*}{\partial w_2} = -p^{\frac{1}{1-a-b}} \left(\frac{a}{w_1}\right)^{\frac{a}{1-a-b}} \left(\frac{b}{w_2}\right)^{\frac{1-a}{1-a-b}} = -z_2^*(w_1, w_2, p)$$

Leme du Hotteling.

2) $F(q_1, q_2, q_3) = \max(0, q_1)^2 + \max(0, q_2)^2 + \min(0, q_3)$

$$F = \begin{cases} q_1^2 + q_2^2 + q_3 & \text{caso } q_1 \geq 0,\ q_2 \geq 0 \text{ e } q_3 \leq 0 \\ q_2^2 + q_3 & \text{caso } q_1 < 0,\ q_2 \geq 0 \text{ e } q_3 \leq 0 \\ q_1^2 + q_3 & \text{caso } q_1 \geq 0,\ q_2 < 0 \text{ e } q_3 \leq 0 \\ q_1^2 + q_2^2 & \text{caso } q_1 \geq 0,\ q_2 \geq 0 \text{ e } q_3 > 0 \\ q_3 & \text{caso } q_1, q_2 < 0 \text{ e } q_3 \leq 0 \\ q_2^2 & \text{caso } q_1 < 0\ \ q_2 \geq 0 \text{ e } q_3 \geq 0 \\ q_1^2 & \text{caso } q_1 \geq 0\ \ q_2 < 0 \text{ e } q_3 > 0 \\ 0 & \text{caso } q_1, q_2 < 0 \text{ e } q_3 = 0 \end{cases}$$

$$\frac{\partial F/q_1}{\partial F/\partial q_3} = \frac{p_1}{p_3} \Rightarrow \frac{2q_1}{1} = \frac{p_1}{p_3} \Rightarrow \boxed{q_1^* = \frac{p_1}{2p_3}}$$

$$\frac{\partial F/\partial q_2}{\partial F/\partial q_3} = \frac{p_2}{p_3} \Rightarrow 2q_2 = \frac{p_2}{p_3} \Rightarrow \boxed{q_2^* = \frac{p_2}{2p_3}}$$

$$\boxed{F(q_1, q_2, q_3) = 0}$$

$$\boxed{q_1^2 + q_2^2 + q_3 = 0} \Rightarrow \boxed{q_3 = -q_1^2 - q_2^2}$$

$$\boxed{q_3^* = -\frac{p_1^2 + p_2^2}{4p_3^2}}$$

$$\pi^* = p_1 q_1^* + p_2 q_2^* + p_3 q_3^* = \frac{p_1^2}{2p_3} + \frac{p_2^2}{2p_3} - \frac{p_1^2 + p_2^2}{4p_3} = \frac{p_1^2 + p_2^2}{4p_3}$$

$$y_1^*(p_1, p_2, p_3) = \frac{p_1}{2p_3} \qquad y_2^*(p_1, p_2, p_3) = \frac{p_2}{2p_3} \qquad y_3^* = -\frac{p_1^2 + p_2^2}{4p_3^2}$$

$$\pi_1^*(p_1, p_2, p_3) = \frac{p_1^2 + p_2^2}{4p_3} \leftarrow$$

$$\frac{\partial \pi_1^*}{\partial p_1} = \frac{2p_1}{4p_3} = \frac{p_1}{2p_3} = y_1^*(p_1, p_2, p_3) \qquad \frac{\partial \pi^*}{\partial p_2} = \frac{2p_2}{4p_3} = \frac{p_2}{2p_3} = y_2^*(p_1, p_2, p_3)$$

$$\frac{\partial \pi_1^*}{\partial p_3} = -\frac{p_1^2 + p_2^2}{4p_3^2} = y_3^*(p_1, p_2, p_3)$$